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The Generation and Modelling of Resultant Tones

Abstract

This report explores the auditory illusion of Resultant Tones through the collation, generation and analysis of documented theoretical methodologies that arithmetically determine resultant tone frequencies. Elements investigated include the analysis of frequency relationships within the resultant tone generation and similar arithmetically determined frequencies present in the physical generation of sound; and the manipulation of physical parameters for emphasising the perception of resultant tones.

Introduction

In the sonic realm, the perception of sound is dependent on the limitations of the sensory capacity. Although perceptions are individual, they are predominantly general interpretations resembling the physical properties of the sound signal. Auditory illusions exploit the limitations of sensory capabilities to create individual perceptions alternative to the physical signal. With the exploration of the limitations of human sensory perception through the auditory illusion *resultant tones*, the elements encompassed in this report are:

- The collation of theoretical methodologies (thirteen in total) detailing the perceptions of *resultant tones*.
- Exploration of the methodologies theoretically and structurally investigating relationships between the methodologies and other structural elements to find a general consensus or element accounting for the frequencies perceived.
- Modelling of the analysed methodological *resultant tones* (the *perceptions*) for physical spectral analysis and comparison to the physical model (the *illusion*).

Terminology

Auditory Illusions

Auditory illusions, as with other sensory illusions, result from stimuli that operate at the extremities of sensory capacity. The stimuli distort sensory perception and create perplexity within the auditory mechanism, resulting in a perceivable difference to the physical composition of the sound. An attempt to clarify the ambiguity between human perception, and the quantifiable physical world, results in further uncertainty as to the actual nature of the sound.

An auditory illusion can be classified into two categories based on the structure of the stimuli:

- Psychoacoustic auditory illusions
- Psychophysic auditory illusions

Psychoacoustic auditory illusions are complex sounds physically constructed to create frequency, spectral, rhythmic and dynamic ambiguity. This can be exemplified by the tonal structures of *Shepard Tones* (Shepard 1964) or the rhythmic configuration of the *Risset Pattern* (Risset 1972).

Psychophysical auditory illusions are predominantly approached from the psychological environment examining sensory strengths and not intensely focused on the complex physical structure of the sound. Rather, they focus on the alternative perceptual response to sound targeted at the extremes of auditory sensory capacity. Superficially, the psychophysical illusion may not be as physically perplexing as the psychoacoustic illusion. However, the psychophysical illusions are still classified as an auditory illusion as they exploit the auditory senses' ability to generate an alternate perception to the physical composite of the sound. Examples of the Psychophysical auditory illusion include those developed by Diana Deutsch, such as the *Octave Illusion* (http://psy.ucsd.edu/~ddeutsch 2005), and *Resultant* (Summation and Difference) *Tones*.

Fundamental frequencies

The term *fundamental frequency* for the purpose of this report represents a frequency employed to generate *resultant tones*. The term is not in reference to harmonically related frequencies; these are referred to as harmonic partials.

Resultant Tones

Resultant tones are the additional tones perceived when combining two or more *fundamental frequencies*. They sound similar to harmonics influencing timbrel signatures through perceptual recognition rather than physical recognition. However, unlike harmonics, *resultant tones* are not dependent on a complex waveform for their generation, nor produced externally to the ear.

Resultant tones go by a number of other terms including *Combination tones, Summation and Difference tones, Third Tones, Tartini Tones* and *Grave Harmonics*. For the purpose of this report, the term *Resultant*

Tone is employed as it encompasses the numerous classes of categorisation.

Environmental Parameters

The environment that the *resultant tones* were generated consisted of the following criteria:

- Headphones for direct and restriction of sound.
- Just Intonation tuning system.
- Sine tone waveform.
- The following three Fundamental frequencies generate the illusion:
  - o 130.8128 C3.
  - o 196.2192 G3 (perfect fifth above F1).
  - o 327.0320 E4 (tenth above F1).

Technical Details

Resultant tones can be arithmetically determined using various mathematic formulas. A variety of methods are used to calculate the frequencies of the additional tones, each adopting different formulas, standards, and classification systems. In many cases however, the equated frequency is the same to the fourth decimal place. Primarily, resultant tones are defined using two arithmetic methods:

Difference tone

The difference tone (PD) is the difference between the two fundamental frequencies calculated by subtracting *fundamental frequency* 1 (F1) from *fundamental frequency* 2 (F2).

F2	-	F1	=	PD
660Hz E5	-	440Hz A4	=	220Hz A3

Example 1. *Difference tone equation*

Summation tone

The summation tone (PS) is the sum of the two *fundamental frequencies* calculated by adding *fundamental frequency* 1 (F1) to *fundamental frequency* 2 (F2).

F1	+	F2	=	PD
440Hz A4	+	660Hz E5	=	1100Hz A3

Example 2. *Summation tone equation*

Multiplied tones

The summation and difference tones are further subdivided into sections based on the minimum of one frequency in the equation multiplied.

[F1 X 2]	-	F2	=	PDM
[440Hz A4 X 2]	-	660Hz E5	=	220Hz A3

Example 3*. Multiplied difference tone equation*

F1	+	[F3	X 2]	=	PSM
440Hz A4	+	[550Hz C5	X 2]	=	1540Hz E6

Example 4. *Multiplied summation tone equation*

The multiplied category is utilised in this report to condense the numerous individual formulas and categorical systems equating the same frequency, to one universal equation.

The examples given so far are further classified as *Primary Resultant Tones*, as the tones are determined using physical frequency in the equations. The employment of physical frequency in an equation is a universal element for the classification of primary or first order tones by all researchers. Secondary and further orders of classification are reliant on individual formulas employed to calculate the frequency. Since numerous equations, methodologies and categorical systems may be used to equate similar outcomes, *Secondary Tones* are defined in this report as tones generated utilising *resultant tones* in the equation.

This research is based on three fundamental frequencies generating twenty-two resultant tones that are divided into nine categorical equation classifications.

Instrument	Classification	Equation
i1	Fundamental (F1)	Frequency input
i2	Fundamental (F2)	i1 + Perfect 5th
i3	Fundamental (F3)	[i1+ i2] x 0.5
i4	Primary Difference (PD)	i2 - i1
i5	Primary Difference (PD)	i2 - i3
i6	Primary Summation (PS)	i1 + i2 + i3
i7	Primary Summation (PS)	i1 + i2
i8	Primary Summation (PS)	i1 + i3
i9	Primary Summation (PS)	i2 + i3
i10	Primary Difference Multiplied (PDM)	[i1 x 2] - i2
i11	Primary Difference Multiplied (PDM)	[i1 x 4] - i2
i12	Primary Difference Multiplied (PDM)	[i3 x 2] - i1
i13	Primary Difference Multiplied (PDM)	[i3 x 2] - i2
i14	Primary Summation Multiplied (PSM)	i1 + [i3 x 2]
i15	Primary Summation Multiplied (PSM)	i2 + [i3 x 2]
i16	Primary Summation Multiplied (PSM)	i1+ i2 +[i3 x 2]
i17	Secondary Difference (SD)	i4 - i5
i18	Secondary Summation (SS)	i4 + i5
i19	Secondary Summation (SS)	i11+ i12
i20	Secondary Summation (SS)	i11 + i14
i21	Secondary Difference Multiplied (SDM)	[i1 x 3] - [i2 x 2]
i22	Secondary Difference Multiplied (SDM)	[i1 x 4] - [i2 x 3]
i23	Secondary Summation Multiplied (SSM)	i7 + i8
i24	Secondary Summation Multiplied (SSM)	i11+ i13
i25	Secondary Summation Multiplied (SSM)	i11+ i15

Table 1. *A list of the classification, instruments and equations used in this research.*

I. Generation of Resultant Tones

The source and generation of *resultant tones* is one of contention as *resultant tones* can be indistinguishable from other sine tone transients.

The dominant theory for perceived tones are frequencies generated within the cochlea of the ear as a result of non-linear distortion. Consequently, *resultant tones* do not appear with the *fundamental frequencies* in the signal when recorded or spectrally analysed.

Three commonly confused phenomena that are considered are neither resultant tones, nor from the source of origin are examined:

Harmonic partials and overtones

Harmonic partials and overtones are dependent on physically complex waveforms, while resultant tones are present in the physically simple sine tones that contain no partials or transients. *Resultant tones* do occur in complex waveforms, in which case the harmonics influence distinguishing the resultant tones. The stronger harmonic partials emphasise the tones by directing the subjects' attention to perceiving all the transients present that may also correspond to *resultant tone* frequencies, thus reinforcing their amplitude. However, this emphasis also inhibits the precise detection of resultant tones. Since *resultant tones* are psychophysically generated transients, they are easily mistaken with the acoustically produced harmonic partials. Therefore, the employment of the sine tone is essential for eliminating harmonic transients and partials.

Sympathetic resonance

Sympathetic resonance is a result of naturally occurring sub and super harmonics partials in a waveform reflecting off objects in space, and is not the source of resultant tones. This is demonstrated with the presence of *resultant tones* in sound transmitted through headphones, thus eliminating spatial influences.

Transmission of sound

The propagation and transmission of sound, such as speaker distortion, produces frequencies similar to the non-linear distorted *resultant tones* produced in the cochlear, however, as *resultant tones*, may be generated on acoustic instruments, it is not the source but an effective tool for reconstructing the perceived *resultant tones* acoustically.

II. Models Generated and Explored

Numerous methods determining individual perceptions of *resultant tones* present in the illusion have been documented regarding the generation source of *resultant tones*, the number of tones present in a sound, the equations that correspond with the frequencies perceived and the relationship of these tones to the *fundamental frequencies*.

After an analysis of the illusion that is common to all perceptions and generating the tones, models and investigations into the methods for determining the frequencies perceived in the following order:

- **II.a Illusion Model:** the physical model that all perceptions are derived
- **II.b 18th Century:** Earlier period of evolution dominated by musician theorists with the following models:
    - Sorge Model
    - Romieu Model
    - Tartini Model
- **II.c 19th Century:** intermediate period of evolution dominated by physics theorists with the following models:
    - Hällström Model
    - Helmholtz Model
- **II.d 20th century:** latter period dominated by physics and psychology theorists with the following models:
    - König Model
    - Krueger Model
    - Titchener Model
    - Stumpf Model
    - Plomp Model
    - Kemp Model
    - Harmonic Series Model

II.a Illusion Model

The illusion is the basis and physical signal that all the documented perceptions are derived from. Physically, it is comprised of the *fundamental frequencies* that generate the *resultant tones* and is the auditory reference for comparison of the simulated perception models. Although many theories specify only two tones are required for the generation of the illusion, the inclusion of the third frequency generates additional *resultant tones*, thus creating a more complex sound.

The *Fundamental frequencies* employed in the illusion model are those determined by i1, i2 and i3. *Refer to table 1 for broader description of the instrument equations.*

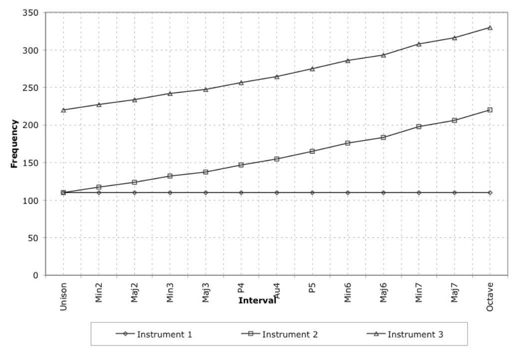

Figure 1*. Illusion Frequency Model*

II.b Sorge Model

German organist Georg Andreas Sorge 1744 theory states that *resultant tones* were the product of beat tones produced externally to the ear (Plomp 1965). Although the beat tone theory does produce sine tone frequencies at calculated distances, these tones like harmonics are not the source of *resultant tones*.

Sorge's frequencies are determined utilising the ratio equation of 3:4:5. *Fundamental frequencies* are generated according to this ratio, with *resultant tones* corresponding to ratio calculations of :1 and :2 (Plomp 1965). In western interval terms, the fundamentals equate to intervals of the 3:4 perfect fourth, 3:5 major sixth, and 4:5 the major third.

The following table expresses the ratio equation in frequency terms and a conversion to the instruments in this research utilising a universal equation.

Tones	Frequency	Instruments
Fundamental 1	440hz (A4)	1, 13, 18, 22
Fundamental 2	586.6667hz (D5)	3 (x 2)
Fundamental 3	733.3334hz (F#5)	2, 11
Difference 1	146.6667hz (D3)	5, 10, 17, 21
Difference 2	293.3334hz (D4)	4

Table 2. *Instruments modelling Sorge's equation*

The difference tones are at the distance of the fifth and twelfth below fundamental 1, forming the partials of 1, 2, 3, 4 and 5 of the harmonic series; the single and double octave below fundamental 2; and the tenth and eighteenth below fundamental 3.

The *resultant tones* employed in *Sorge* 's model are those determined by i1, i2, i3, i4, i5, i10, i11, i17, i21 and i22. *Refer to table 1 for broader description of the instrument equations.*

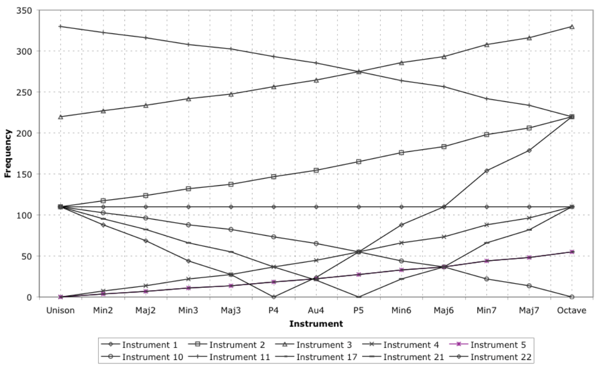

Figure 2 *Sorge Frequency Model*

Romieu Model

French physicist Jean-Baptiste Romieu based his theory, similar to Sorge's, on the ratio relationship of the two *fundamental frequencies*, with difference tone frequency corresponding to the greatest common divisor (Romieu 1758).

Romieu also noted that the tones were only perceived on the pitched intervals of the perfect fourth, perfect fifth and major sixth. However, unlike Sorge who perceived two difference tones simultaneously on these intervals, Romieu only perceived the one tone that, based on his classification system determined it as a primary difference tone resulting from physical sound. Analysis of the frequency equated at the specified intervals corresponds to the universal classification of the frequency as a secondary tone arising from the psychophysical perception of other resultant tones.

The *resultant tones* employed in Romieu's model are those determined by i1, i2, i3, i4, i5 and i21. *Refer to table 1 for broader description of the instrument equations.*

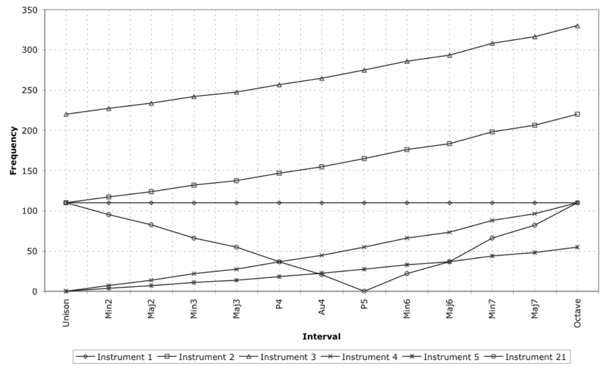

Figure 3. *Romieu Frequency Model*

Tartini Model

Italian violinist Giuseppe Tartini's investigations into the difference tone (Tartini 1754) were based on the sawtooth wave that contains the most harmonic partials of all waveforms and further emphasised the difference tone. As a result, Tartini ascribed to his predecessor's theory that resultant tones were produced externally of the ear as a result of beat tone theory.

The difference tone credited to Tartini as the *Tartini Tone* (commonly PD) is not the actual interval, frequency or the equation documented in the 1754 *Trattato di Musica*. In this treatise he called the phenomena the *Terzo Sona* (Tartini 1754) and employed it as the basis for his tuning system. The equation to calculate his difference tone detailed below:

[F2 - F1 = PD] X 2 = Terzo Sona
[660 E5 - 440 A4 = 220 A3] X 2 = 440 A4

Example 5*. Terzo Sona equation*

The *Terzo Sona* equates to one octave above the primary differential frequency that is the equation for determining the *Tartini Tone* (PD).

Latter scientists did not employ the *Terzo Sona*, however, since Tartini ascribed to his predecessor's theory that the *resultant tones* were interval specific, the *Terzo Sona's* frequency then corresponds to frequencies calculated utilising alternative formulas. These frequencies correspond to the equations employed by i10, i13, i21 and i22 for the intervals previously stated by Sorge and Romieu.

The equations employed in Tartini's model are those determined by i1, i2, i3, i4, i21 and i22. *Refer to table 1 for broader description of the instrument equations*

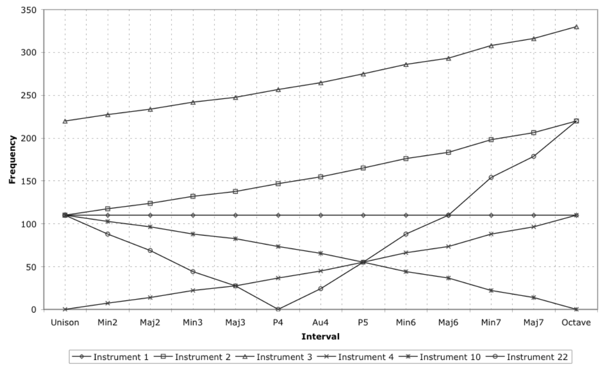

Figure 4. *Tartini Frequency Model*

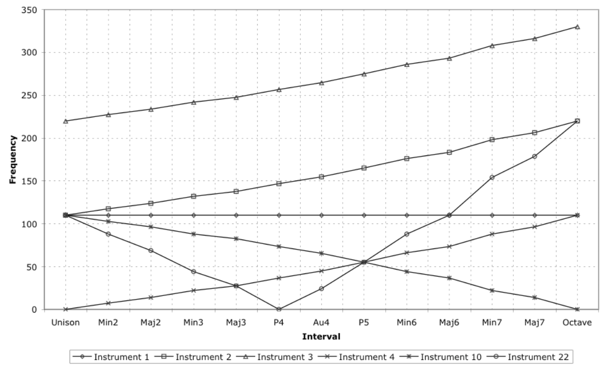

Figure 5. *Hällström Frequency Model*

II.c Hällström Model

German physicist Gustaf Gabriel Hällström published the results from an investigation into *resultant tones* conducted on the violin, emulating Tartini's method of utilising the sawtooth waveform to emphasise his perception of the *resultant tones* (Hällström 1832). On the basis of his findings, Hällström introduces the possibility of secondary generation and classification of difference tones. These are the frequencies not created from physical sound as demonstrated by the primary tones, but rather, generated from the previous phantom type perceptions of *difference tones*.

Hällström classified *resultant tones* according to the frequencies employed in the equations, ranging in first to fourth orders as detailed in the following table and converted to the Universal equation (Hällström 182).

Hällström		Universal Classification	
Order	**Equation**	**Equation**	**Instrument**
1st	I2 – i1 = d1	I2 – 1	4
2nd	I1 - d1 = d2	[i1 x 2] – i2	10
3rd	I2 - d2 = d3	[i1 x 2] – [i2 x 2]	8
4th	d2 - d1 = d4	[i1 x 3] – [i2 x 2]	21

Table 5. *Hällströms equations converted to the Universal Equation*

In the previous table, Hällström's theory demonstrates that first-order *resultant tones* induce the generation of second and further orders of *resultant tones*, hence, suggesting that the subject is not only able to interpret the distortion of one frequency, but many frequencies independently (Hällström 1832). That is unless the actual generation of the *resultant tone* frequencies are calculated using an alternative formula. The alternative formulas as expressed in this report with the equations detailed in the above chart, propose that the frequencies do not occur as independent distortion, but rather from the multiplication of the sound signal as later explored by König, Krueger, Titchener and Stumpf.

The *resultant tones* employed in Hällström 's model are those determined by i1, i2, i3, i4, i10 and i21. *Refer to table 1 for broader description of the instrument equations.*

Helmholtz Model

German physicist Hermann Von Helmholtz introduced the existence of summation tones and proposed that the source and generation of *resultant tones* differed to the acoustically produced harmonics (Helmholtz 1895). This theory is based on his ability to generate the tones using the transient free sine tone tuning forks, and that the tones were occurring inside the ear (previously theorised to be generated in the acoustic environment). Although erroneously credited to 28 independent resonators, the concept is based on the ability to perceive numerous *resultant tones* at any one time and the creation of secondary order tones, that arise not from physical sound but rather, the phantom type perceived *resultant tones*. However, a small amount inconsistency appears in Helmholtz's work, with the existence of the psycho-physically generated secondary orders of difference tones but not with the summation tones. This absence cannot be explained as the higher order difference tones may only theoretically exist, as mentioned by Alexander J. Ellis in the translator's note "the extent and audibility of these had yet to be proven" (Ellis 1895).

Determination of Helmholtz's *resultant tones* (following previous methodologies), are conditional on specific intervals that also dictate the number of tones and orders generated. (Helmholtz 1895).

With the number of orders reaching as far as the sixth, Helmholtz's classification method is examined for comparison with other methods. The equations used to determine Helmholtz's secondary order of frequencies ascribe to Hällström's methodology, that the secondary plus tones are generated from psychophysical *resultant tones*, and not utilising König's method of multiplying physical sound. Closer analysis however reveals that many of Helmholtz's equated frequencies may be encompassed in the multiplied equation classification system, therefore, reducing the number of orders.

The following table illustrates the conversion of Helmholtz's difference tone utilising the ratio equation specified for the minor sixth interval 5:8 (Helmholtz 1895) to the universal equations and classifications used in this report.

Helmholtz		Universal Classification		
Order	Ratio Equation	Order	Equation	Instrument
1	8 - 5 = 3	PD	I2 - i1	4
2.1	5 - 3 = 2	PDM	I1 - i4	10
2.2	8 - 3 = 5	PDM	I2 - i4	13
3.1	5 - 2 = 3	PD	I2 - i1	4
3.2	8 - 2 = 6	-	-	-
4.2	3 - 2 = 1	SDM	[i1 x 3] - [i2 x 2]	21
4.2	5 - 3 = 2	PDM	[i1 x 2] - i2	10
5.1	5 - 1 = 4	-	-	-
5.2	8 - 1 = 7	-	-	-
6.1	8 - 7 = 1	SDM	[i1 x 3] - [i2 x 2]	21
6.2	5 - 4 = 1	SDM	[i1 x 3] - [i2 x 2]	21
6.3	4 - 2 = 2	PDM	[i1 x 2] - i2	10
6.4	8 - 4 = 4	SDM	[i1 x 4] - [i2 x 3]	22

Table 7. *Helmholtz's equation conversion*

The *resultant tones* employed in Helmholtz's model are those determined by i1, i2, i3, i4, i5, i6, i7, i8, i9, i10 and i22. *Refer to table 1 for broader description of the instrument equations*

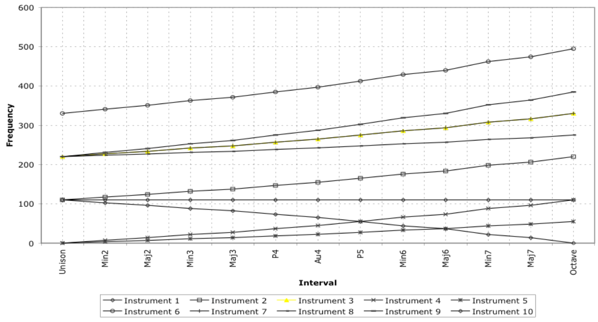

Figure 6. *Helmholtz Frequency Model.*

II.d König Model

German physicist Karl Rudolf König published his results in an article detailing explorations on the beat tone theory and *resultant tones* generated by two sine tones (König 1876). For equating the frequencies, König introduced the factor of multiplying the fundamentals in order to produce the *secondary resultant tones*, rather than the method employed by his predecessors of utilising *resultant tones*.

The *resultant tones* employed in König's model are those determined by i1, i2, i3, i4, i10, i11 and i12. *Refer to table 1 for broader description of the instrument equations*.

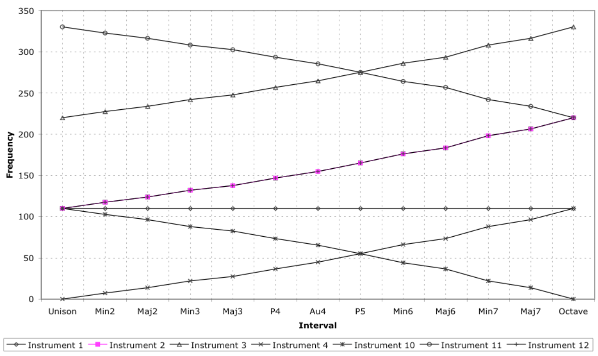

Figure 7. *König Frequency Model.*

Krueger Model

German psychologist Felix Krueger expanded on König's method of multiplying frequencies in the equation (Krueger 1900). The frequencies determined however can be reproduced using the universal equations utilised in this report. The following table illustrates a conversion between Krueger's equation and the Universal:

Krueger Equation	Universal Equation	Instrument
i1 - i4 = diff2	[i1 x 2] - i2	10
Diff2 – diff1 = diff3	[i1 x 3] - [i2 x 2]	21
Diff2 – diff1 = diff3	[i1 x 4] - [i2 x 3]	22

Table 8. *Krueger equation conversion*

The *resultant tones* employed in Krueger's model are those determined by i1, i2, i3, i4, i10, i21 and i22. *Refer to table 1 for broader description of the instrument equations*.

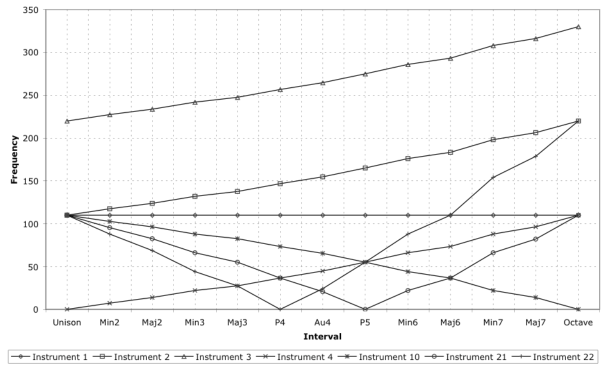

Figure 8. *Krueger Frequency Model.*

Titchener Model

English psychologist Edward Bradford Titchener's methodology employs the primary difference and summation tones and plus an additional new secondary difference tones as equated by i17 and i18 (Titchener 1905).

The *resultant tones* employed in Titchener's model are those determined by i1, i2, i3, i4, i7, i10, i11 and i21. *Refer to table 1 for broader description of the instrument equations*.

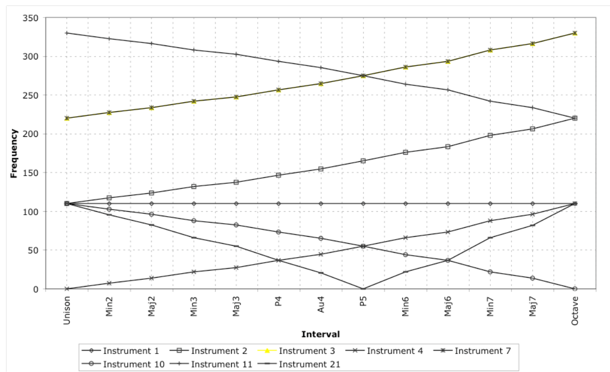

Figure 9. *Titchener Frequency Model.*

Stumpf Model

German acoustician Carl Stumpf duplicated Krueger's experiments over a larger frequency range, analysing the effect of fundamental intervals on the perception and frequency of *resultant tones* (Stumpf 1910). The results indicate only the frequencies corresponding with i4 and i10 are strongly perceived when the interval between the *fundamental frequencies* expands above the octave, and the presence of other resultant tones are faintly perceived when the interval lies below the octave. (Stumpf 1910).

The *resultant tones* employed in Stumpf's model are those determined by i1, i2, i3, i4, i5, i6, i7 and i10. *Refer to table 1 for broader description of the instrument equations.*

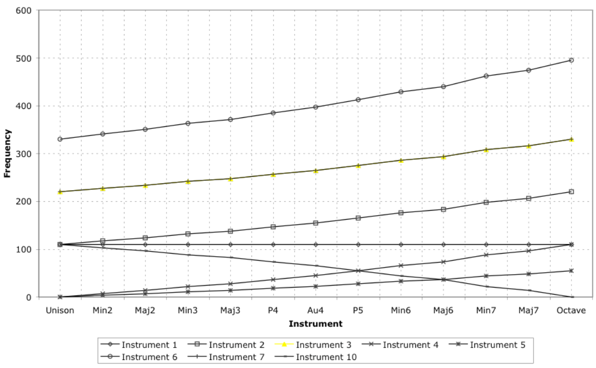

Figure 10. *Stumpf Frequency Model.*

Plomp Model

Reiner Plomp determined the *resultant tones* perceived after publishing his detailed historical and theoretical article on previous research conducted (Plomp 1965).

The *resultant tones* employed in Plomp's model are those determined by i1, i2, i3, i4, i5, i10 and i12. *Refer to table 1 for broader description of the instrument equations.*

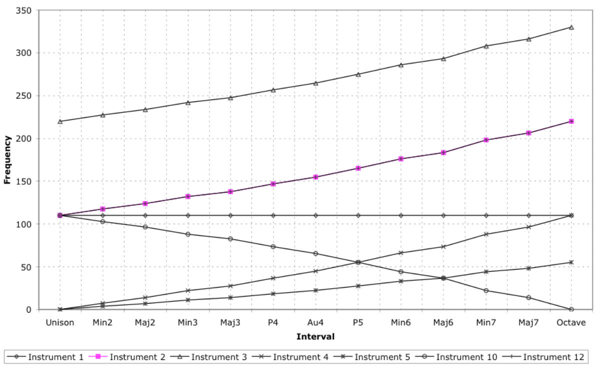

Figure 11. *Plomp Frequency Model.*

Kemp Model

Dr. David Kemp furthered Helmholtz's theory, that the source of *resultant tone* generation was internal of the ear by introducing the concept that the tones were generated in the cochlear as a result of non-linear distortion from otoacoustic emissions (Furst, Rabinowitz and Zurek 1988).

Whilst it is the current popular theory of non-linear distortion, accrediting the generation to otoacoustic emissions is a misnomer as it is only applicable to i4 equation and doesn't account or include the vast range of other frequencies perceived and documented.

The *resultant tones* employed in Kemp's model are those determined by i1, i2, i3, i4 and i10. *Refer to table 1 for broader description of the instrument equations.*

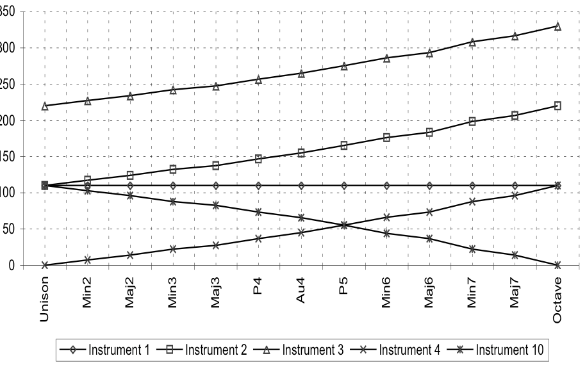

Figure 12. *Kemp Frequency Model.*

Harmonic Series Model

Investigations into the previous models have influenced the analysis and generation of *resultant tone* frequencies based on the harmonic series structure.

The structure is based on the two fundamentals at the perfect fifth interval, and a third instrument at the median x 2. From this simple intervallic relationship, further *instrument* frequencies were generated corresponding to the harmonic series partials.

Although the harmonic series model includes a considerable amount of frequencies not strictly falling into the harmonic series structure, these frequencies exhibit strong harmonic relationships to the both the *fundamental frequencies* involved in generating the illusion, and partials structured in the harmonic series.

The amount of repetitive frequencies and their relative amplitude are reduced to keep the illusion intact. This relates to the frequencies corresponding with the partials 2, 3 and 5, and therefore doubling their amplitude and emphasis.

The following table details the Harmonic Series structure, partials and frequencies corresponding to instrument equations.

Partial	Note name	Instrument
-	C1	5, 17
1	C2	4, 10
2	C3	**1**
3	G3	**2**
4	C4	-
5	E4	**3**
6	G4	-
7	Bb5	13
8	C5	15
9	D5	19
10	E5	16
11	F#5	20
12	G5	24
13	A5 flat	25

Table 10*. Harmonic series structure*

The *resultant tones* employed in the Harmonic Series model are those determined by i1, i2, i3, i4, i5, i6, i7, i10, i12, i13, i14, i15, i16, i17, i19, i20, i23, i24 and i25. *Refer to table 1 for broader description of the instrument equations.*

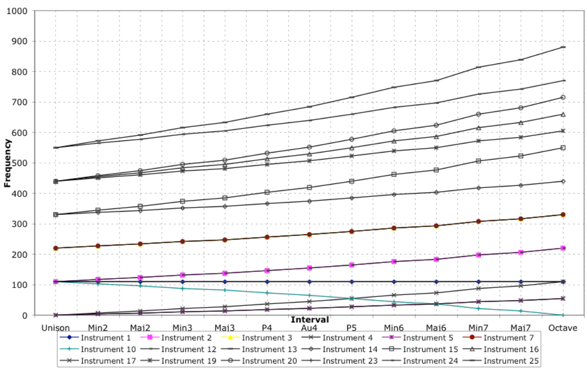

Figure 13. *Harmonic series structure*

Equation Summary

In many instances, the documented individual equations detailing the frequencies of the *resultant tones* perceived often correspond to those recognised by other scientists using alternative equations. This suggests that the frequencies perceived are at common intervals dependent on strong relationships to the *fundamental frequencies*.

Additionally, the one difference tone equated by i4 is the only *resultant tone* detected in all perceptions, suggesting it's dominance as the strongest and easiest perceivable *resultant tone.*

III. Additional parameters for generation

Additional elements applied to the sound signal that assists in the successful creation and perception of *resultant tones*, or alternatively manipulable to deconstruct the illusion, include the following elements:

- Fundamental frequency waveform
- Fundamental frequency intervals
- Fundamental frequency register
- Fundamental frequency motion
- Tuning systems
- Amplitude
- Duration

Fundamental frequency waveform

The structure of the waveform employed for the *fundamental frequencies* is essential for the correct perception of frequency and number of *resultant tones* present in the signal. Distinguishing *resultant tones* from harmonic partials present in the waveform can be difficult as they are both sine tone transients at elegant relationships to the *fundamental frequency*, and at times corresponding at the same interval.

The waveforms explored for this report include the sawtooth, triangle, square and sine tone. The waveform analysis allows for the exploration of partials and harmonics within the waveforms for successful generation of the illusion, or alternatively, deconstruction to enable a greater conception of the illusion.

The sawtooth is the original waveform generated from the violin utilised in the exploration of resultant tones by Tartini (Tartini 1754) and Hällström (Hällström 1832). This waveform provides a complex harmonic spectrum with the associated partials that influence *resultant tones.* The sawtooth wave generates beat tones that, additionally combined with the harmonic partials, makes it difficult to differentiate the *resultant tones* from the harmonics and beat tone transients.

The square and triangle waves are moderate variations of the sawtooth with harmonics corresponding to the odd and even partials of the spectrum respectively. As such, the square and triangle produce similar effects on the perception, however due to the alternative partials, at a lesser impact.

The sine tone was initially explored through the tuning fork as the basis for Helmholtz's (Helmholtz 1895) and König's (Plomp 1965) *resultant tones*. Since the sine tone does not generate harmonics partials, the additional frequencies perceived are *resultant tones* and therefore, the sine tone is essential for clinical testing and precise determination of *resultant tones.*

Fundamental frequency intervals

The *fundamental frequency* intervals that generate the strongest *resultant tones* have been documented in depth by all of the theorists, ranging in the western pitch intervals from minor second to perfect octave.

The interval determined through this research that generates the strongest perception of *resultant tones* is the perfect fifth. This finding supports previous documentation from Helmholtz (Helmholtz 1895), Sorge (Plomp 1965), and Romieu (Romieu 1758). The illusion may be further enhanced by adding an additional frequency at the median of the two frequencies multiplied by 2 (corresponding to an octave higher).

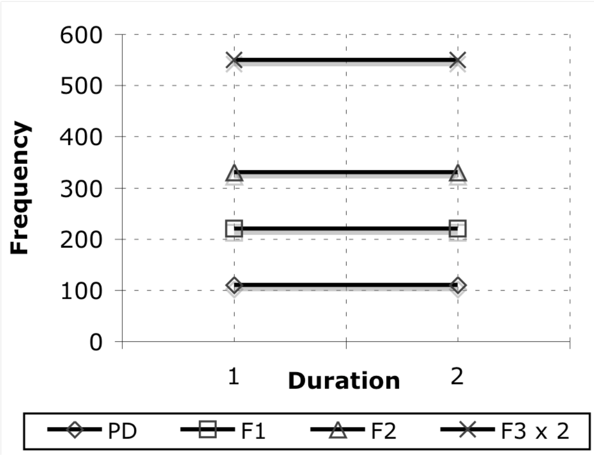

Figure 14. Frequency intervallic relationships

An explanation for the emphasis created from the added third, is that F3 is of close harmonic relationship with F1 and F2, all multiples of the PD (110Hz), and further forming partials 1, 2, 3, and 5 of the harmonic series.

Other strong intervals for generating *resultant tones* are the perfect fourth, minor third, major third, minor sixth and major sixth, thus supporting experiments documented by Sorge and Romieu. (Plomp 1965).

Fundamental frequency register

The register of the *fundamental frequencies* is imperative to the successful generation of the illusion. The higher in register the *fundamental frequencies* rise, the *difference tones* rise accordingly and becomes greater and easier to perceive. Alternatively, the higher the summation tones rise, the greater the difficulty of distinguishable frequency perception.

Although previous experiments were conducted around 520Hz (Titchener 1901) and 440Hz (Helmholtz 1895), the research conducted for this report indicated that *fundamental frequencies* pitched above 440Hz began to destroy the illusion. This occurred through the fragmentation of the individual frequencies as the distance between each frequency increased, thus destroying the overall fluidity of the illusion. The *fundamental frequency* that generates easiest perceived resultant tones with difference and summation tones at easily distinguishable frequencies and an overall smooth illusion is based on a F1 frequency of 130.8128Hz.

Fundamental frequency motion

This research is based on stationary intervals between the *fundamental frequencies* to construct a stable consensus for the comparison of elements explored. However, as demonstrated in the following graph, the *resultant tones* move in response to the *fundamental frequency* motion.

To clarify, the following graph depicts i1 stationary on 110Hz with i2 ascending from the perfect fifth below i1 to the perfect fifth above. This rise causes PD i4 to move accordingly in frequency.

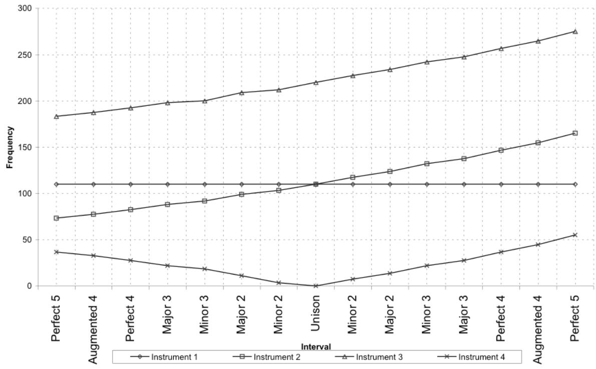

Figure 15. F2 intervallic manipulation

The exploration of tonal systems

The equal tempered and just intonation tuning systems are explored for the purpose of examining the frequency relationships of the *resultant tones* to the *fundamentals* and the effects generated. The concept of employing just intonation is derived from the original ratio method for calculating *resultant tones* utilised by Sorge, Romieu, Tartini and Helmholtz.

Aesthetically, the equal tempered system will generate beat tones, as the frequencies are not in a close harmonic relationship.

The just intonation system structures the frequencies according to the ratio of the interval selected. As such, the harmonic continuity eliminates beat tones.

A comparison of the two tuning systems for *resultant tone* perception shows that the equal tempered system, with slight detuning and beat tones, allows for superficially clearer detection of all transients. However, the just intonation tuning system, with ratio equated harmonic elegance, allows for difficult but precise detection of *resultant tones*.

Amplitude

The generation of a successful illusion requires the precise levelling of amplitude at two levels:

- Overall amplitude.
- Comparative amplitude between the *fundamentals* and *resultant tones*.

As documented by Helmholtz, the overall amplitude needs to be sufficiently loud to perceive *resultant tones*. However, the instrument generating Helmholtz's tones was an intrinsically soft tuning fork mounted on a resonating box (Helmholtz 1895). Therefore, the suggestion of volume needs to be taken into the context of the instrument and era, as the notion of sufficient loudness would differ by current standards. Another explanation for Helmholtz's suggested level is for tonal consistency through stability. Stability occurs from the strength and projection of tone, rather than actual volume. Further, if the subject cannot easily distinguish *resultant tones* then raising the volume for clarification overwhelms the softer tones. This argument supports Stumpf research stating "very high loudness levels are not favourable to distinguish faint combination tones" (Stumpf 1910).

The degree of contrast between the amplitude levels of the *fundamental*s and the *resultant tones* remains undocumented. Since the *resultant tones* are generated from *fundamental frequencies*, the respective amplitude level will be softer. Consequently, this raises a number of considerations. Firstly, *resultant tones* should be hierarchical and not of the same amplitude. This notion is based on an audibility hierarchy where the aforementioned theories of tone perception are interpreted. Secondly, the theory of just noticeable difference should be considered for the successful generation of the illusion.

Therefore, the determined amplitudes of *resultant tones* to *fundamental frequencies*, taking into account the previous elements of hierarchy and just noticeable difference, are equated according to the following ratios.

Resultant tone	Amp ratio RT : FF
Primary difference tones	5:30
Primary resultant tones	4:30
Secondary resultant tones	3:30

Table 11. *Resultant tone amplitude ratio*

Duration

The duration of the tones for successful perception are predominantly dependent on the previous elements explored, such as the intervals, register and amplitude. Superficially, all of the simulated perception models should sound like the Illusion. However, the *resultant tones* only appear when listened to holistically over at least a five second duration. Further, is entirely dependent on the subject's ability to perceive the tones successfully.

IV. Results and Discussion

The following signal analysis tools were employed to determine the physical and spectral properties of the sound signal, thus determining all transients present in the acoustic generation of the sound. The two signals analysed are the *Illusion* - that is comprised of the *fundamental frequencies*, and the *Perception* that is comprised of the *fundamental frequencies* plus the modelled *resultant tones* physically added.

Oscilloscope Analysis

The oscilloscope displays the wave shape according to the combined frequency of the signal as physically sampled. An element that would be visible if the *resultant tones* were present physically is the additional cycles corresponding to the frequency.

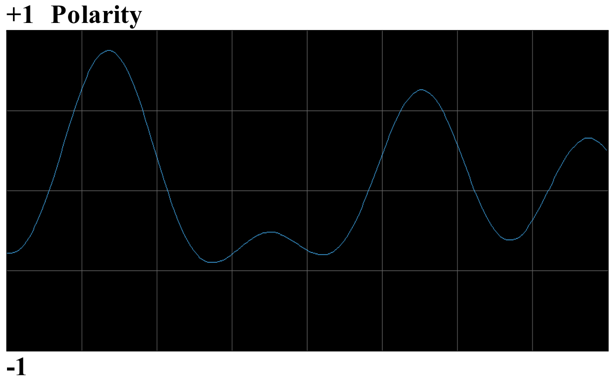

Figure 16. *Illusion wave shape*

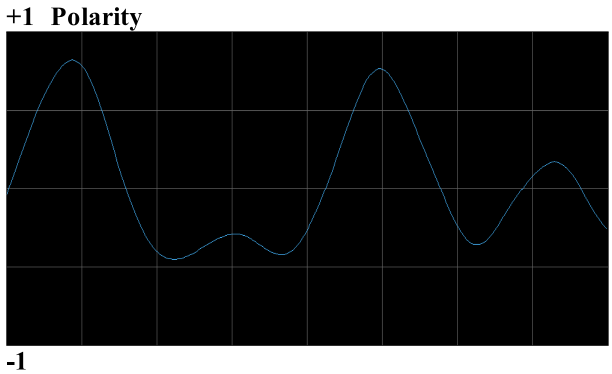

Figure 17. *Perception Model wave shape*

The perception model physically adds the *resultant tones* to the sound signal, and surprisingly, there is barely any difference in wave shape between the illusion and the perception oscilloscope windows. An explanation for the similarity between the two signals lies with many of the *resultant tone* frequencies being of strong relationship to the *fundamentals*.

Fast Fourier Transform analysis

Performing a Fast Fourier Transform analysis on the sound signal arithmetically detects all frequencies, including harmonics and other transients present in the physical sound.

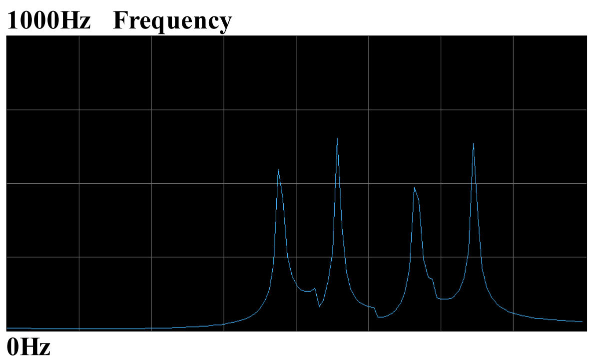

Figure 18. *Illusion Fourier analysis*

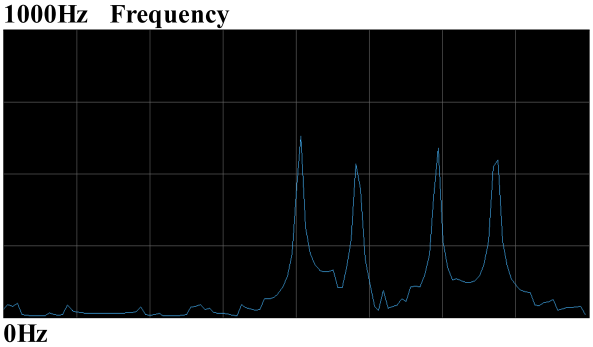

Figure 19. *Perception Fourier analysis*

The Fourier analysis of the illusion signal detects the dramatic peaks corresponding to the *fundamental frequencies*, surrounded by an otherwise smooth shape. Alternatively, the perception signal displays many small peaks depicting the *resultant tones* that are physically added to the signal. A comparison of the two windows indicates that *resultant tones* are not physically present in the illusion signal.

Sonographic analysis

The spectral analysis of the sound signal using a sonogram is employed to detect all possible frequencies present in the physical signal of the acoustic environment. The detected frequencies are displayed in bands at the corresponding frequency.

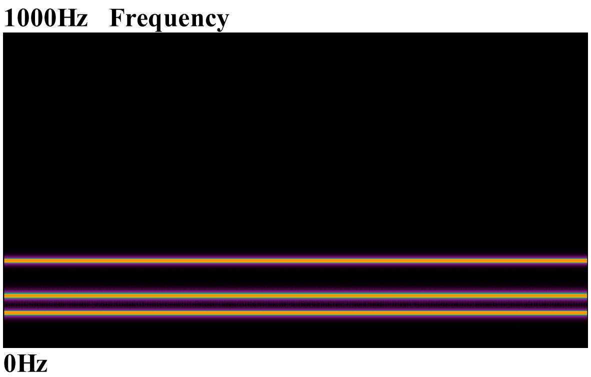

Figure 20. *Illusion sonographic analysis*

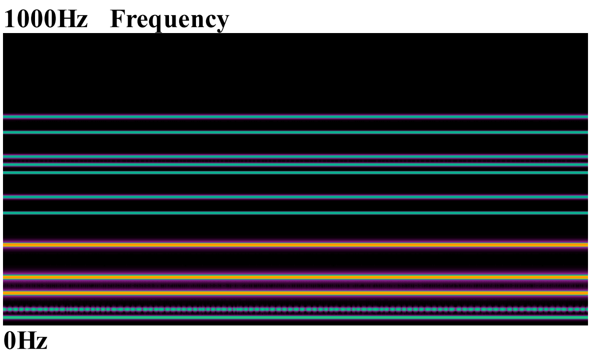

Figure 21. *Perception sonographic analysis*

Analysis of the illusion sonogram depicts only the *fundamental frequencies* and no *resultant tones* in the sound signal.

The perception sonogram displays the model with the *resultant tones* acoustically added, depicting how signal would appear if the *resultant tones* were physically present in the sound signal. A comparison of the two sonograms demonstrates that the *resultant tones* are not acoustically present in the physical signal.

Spectrographic analysis

The spectrograph is an effective tool for the detection of *resultant tones* as it depicts all frequencies and corresponding amplitudes in the physical signal. If a *resultant tone* is present in the physical signal, then this will be displayed in the spectrogram detailing both at the corresponding frequency and amplitude intensity.

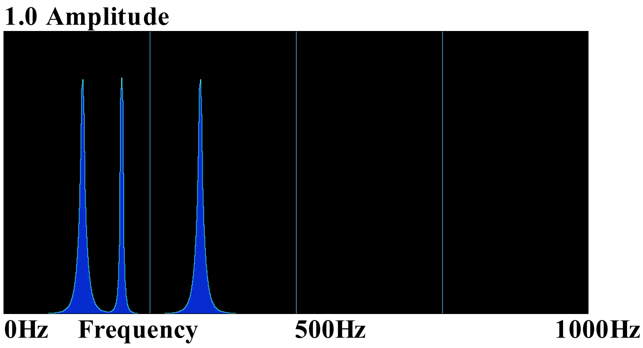

Figure 22. *Illusion spectrographic analysis*

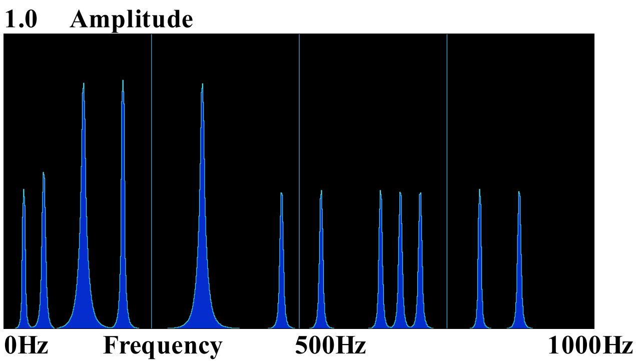

Figure 23. *Perception spectrographic analysis*

The illusion spectrographic analysis displays only the *fundamental frequencies* at the corresponding amplitudes and doesn't detect any *resultant tones* in the sound signal.

The perception spectrographic analysis displays the model with the *resultant tones* physically added to the signal, and depicts how the signal would appear if the *resultant tones* were physically present in the sound signal.

A comparison of the two spectrograms reinstates that none of the *resultant tones* that appear in the simulated perceptual model are represented in the illusion spectrogram and are therefore not present in the physical signal.

V. Conclusion

The research into the perception of *resultant tones* has been conducted through the collation and exploration of a range of theoretical methodologies.

Investigations into the generation and perception of *resultant tones* reveal common elements in the different methodologies. The results indicate that the individual equations determining the frequencies of *resultant tones* often correspond to the fourth decimal place with those of other theorists. This suggests that the frequencies perceived be at common intervals and dependent on an elegant relationship with the *fundamental frequencies*.

Other arithmetic elements common to the methodologies include interval specific generation of the *resultant tones*. For prime audibility, the *fundamental frequency* intervals are predominantly based on the perfect fifth, followed by the major third, minor third, major sixth and minor sixth. The perception of the illusion may be enhanced through the generation at moderate registers, between 110Hz to 440Hz, thus allowing the *resultant tones* not to be concealed by the *fundamental frequencies* or exposed through intervallic distance. Also a moderate level of amplitude is sufficient for detecting *resultant tones*, discounting the theory that the tones require a significant volume, explainable via periodic and instrumental context.

Further analysis of the *resultant tones* indicated that the frequencies consist of a simple arithmetic relationship to the fundamentals and interrelations with other *resultant tones*. These frequencies, on numerous occasions, also correspond with other important arithmetic relationships, such as harmonic series partials and beat tones that may inhibit the accurate perception of *resultant tones*. To determine the precise perception of *resultant tones*, the waveform used to generate the tones must be free of the harmonics, beat tones and other transients that are present in complex waveforms. This was only possible with the implementation of a sine tone.

Inspection of the analyses confirms that, apart from the Oscilloscope, whose analysis of the physical signal is inconclusive, the Fast Fourier Transform, Sonogram and Spectrogram clearly do not detect *resultant tone* frequencies present in the physical signal. Hence, are produced psychophysically as a result of stimulus exploitation at the extremities of human auditory perception.

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